

# Automorphic string amplitudes

Henrik Gustafsson

SITP Seminar  
Stanford 2016

 [hgustafsson.se](http://hgustafsson.se)

# Based on

*Small automorphic representations and degenerate Whittaker vectors*

HG, Axel Kleinschmidt, Daniel Persson

[arXiv:1412.5625](https://arxiv.org/abs/1412.5625) [math.NT]

[GKP14]

Journal of Number Theory 166 (Sep, 2016) 344–399

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$E_6, E_7, E_8$

# Outline

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- Scattering amplitudes  
4-graviton | Derivative expansion | U-duality | SUSY constraints

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- Main results

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- Outlook

# Motivation

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- Langlands program  
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Two-dimensional models of crystals

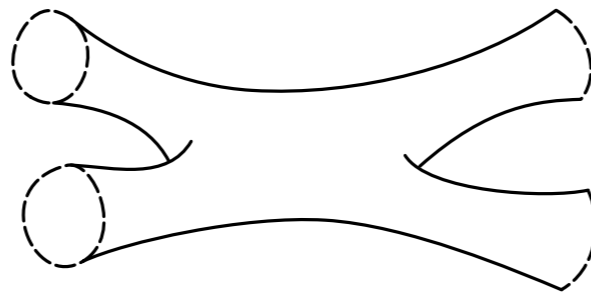
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# String theory

Toroidal compactifications of type IIB string theory

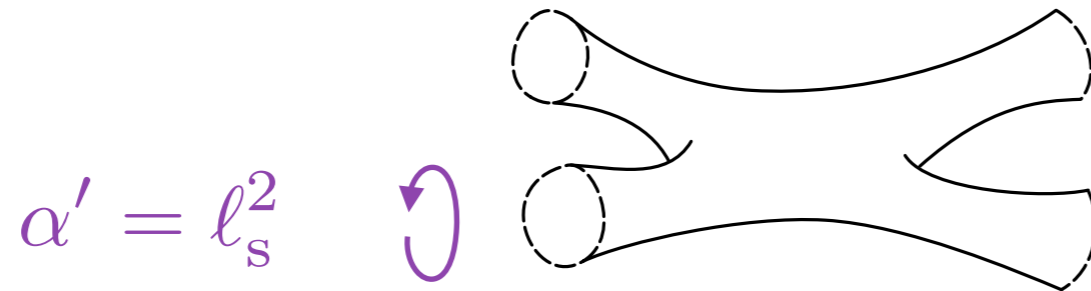
4-graviton scattering amplitudes



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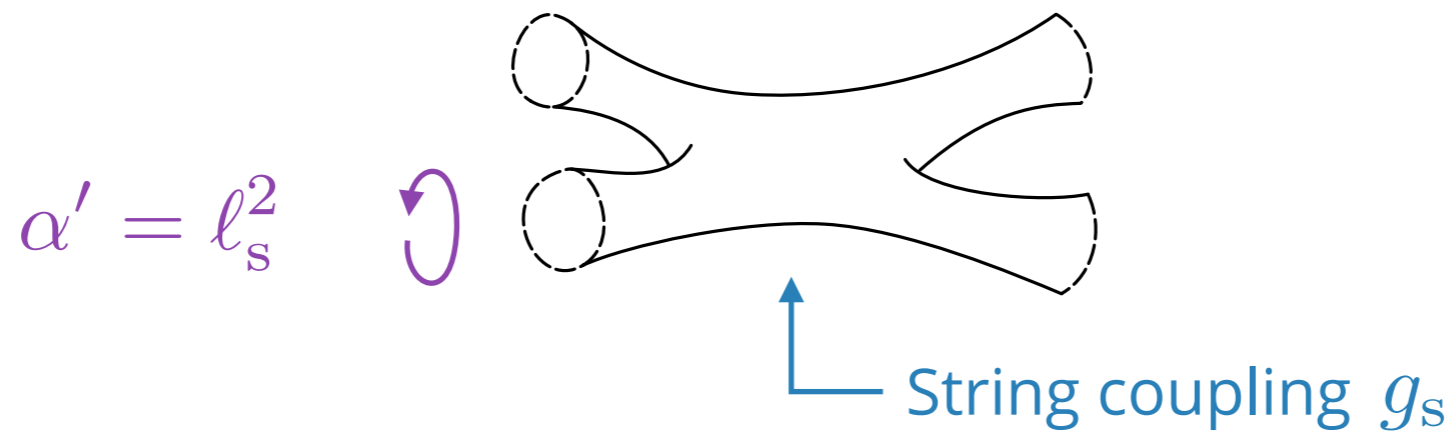




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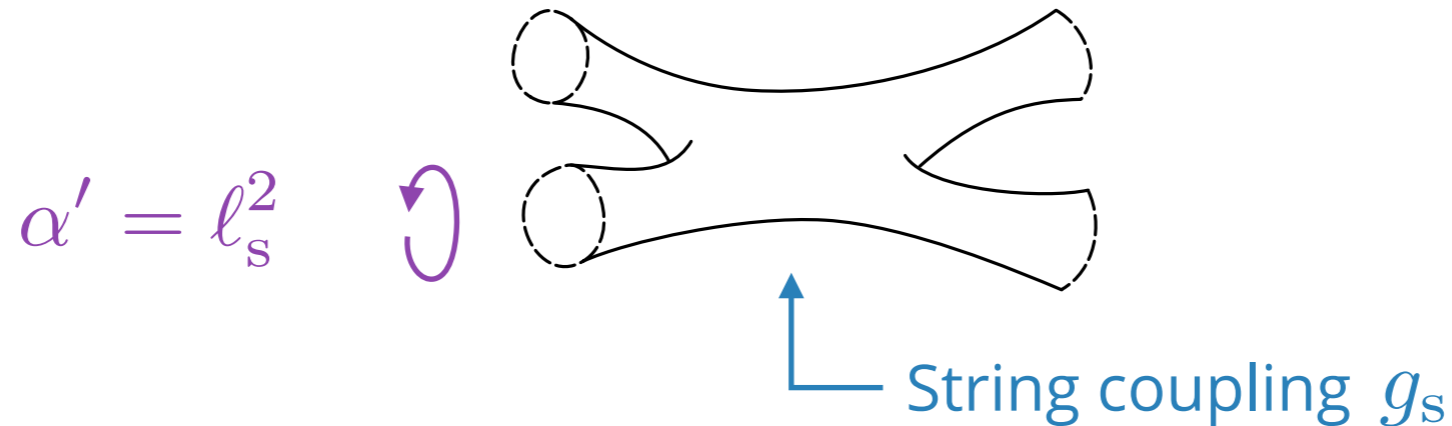
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Toroidal compactifications of type IIB string theory

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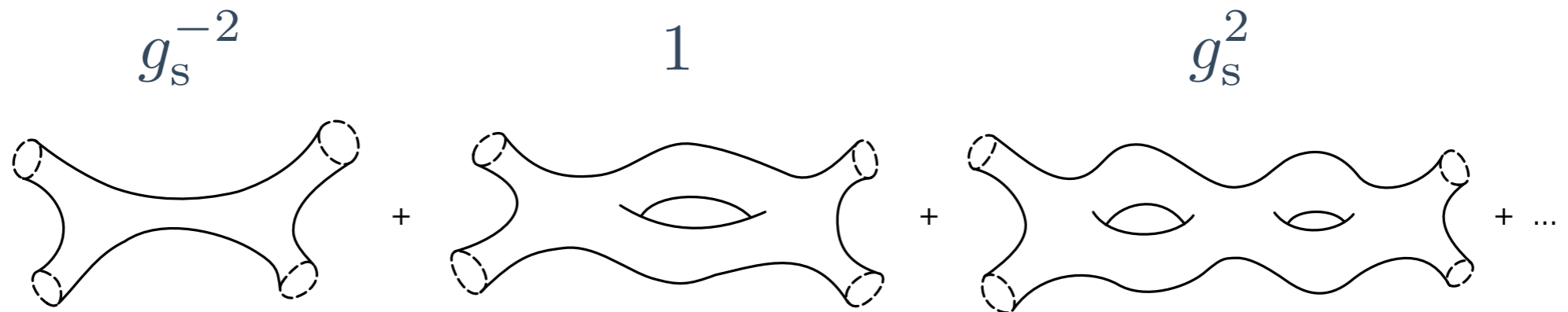


$$s = -\frac{\alpha'}{4}(k_1 + k_2)^2$$

$$t = -\frac{\alpha'}{4}(k_1 + k_3)^2$$

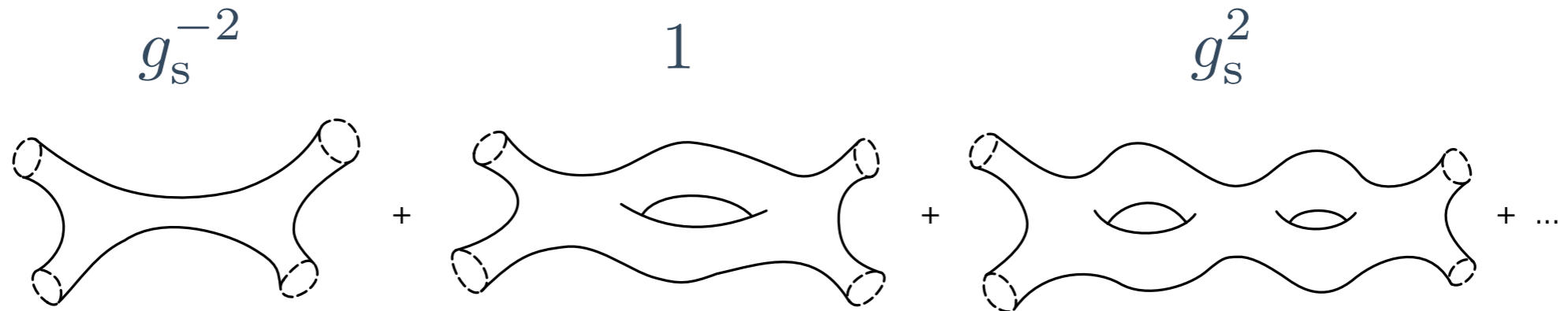
$$u = -\frac{\alpha'}{4}(k_1 + k_4)^2$$

# Interactions



4-graviton amplitude in 10 dimensions:

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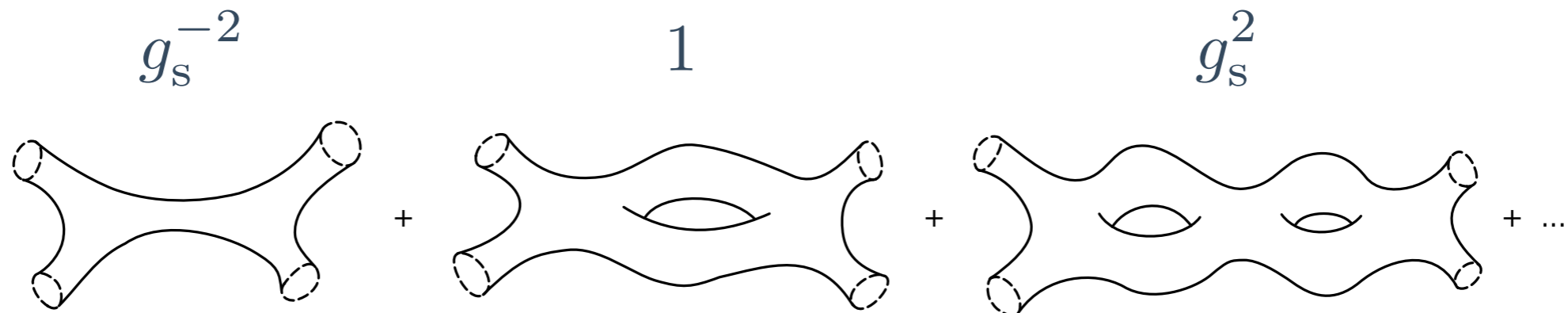


4-graviton amplitude in 10 dimensions:

$$\mathcal{A} = \left( g_s^{-2} \frac{1}{stu} \frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} \right) \mathcal{R}^4$$

[Green-Schwarz, Green-Schwarz-Brink, Gross-Witten]

# Interactions



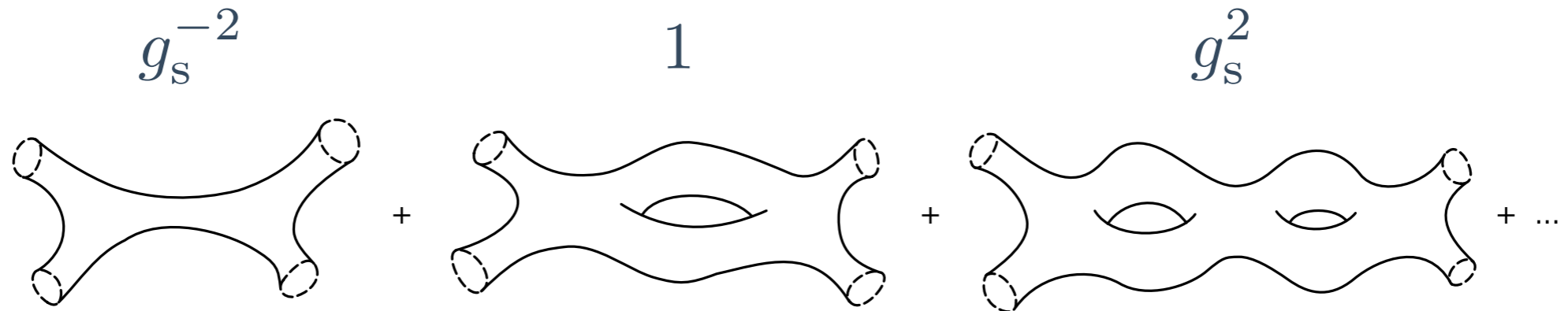
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Kinematic term:  
 Contraction of 4 linearized  
 Riemann tensors  
 $t_8 t_8 \mathcal{R}^4$

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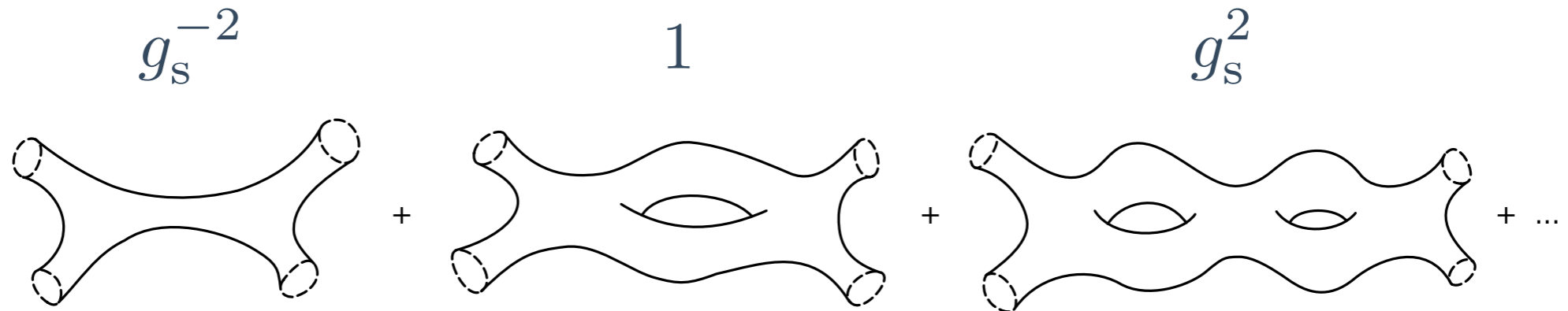


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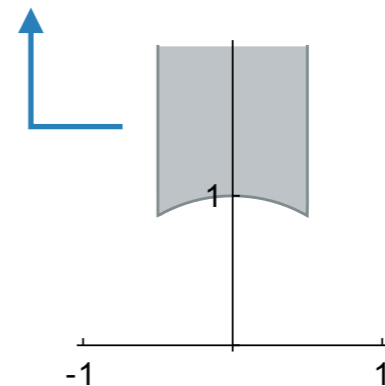
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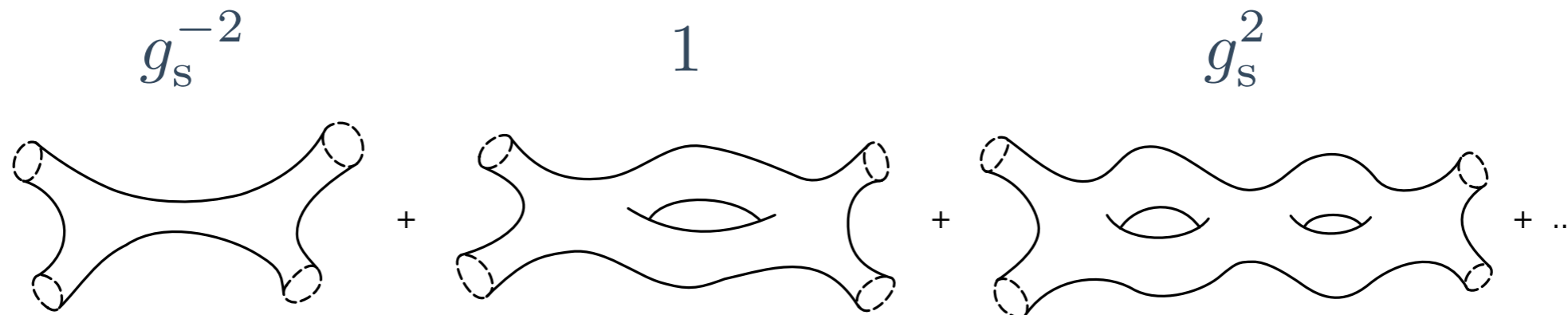
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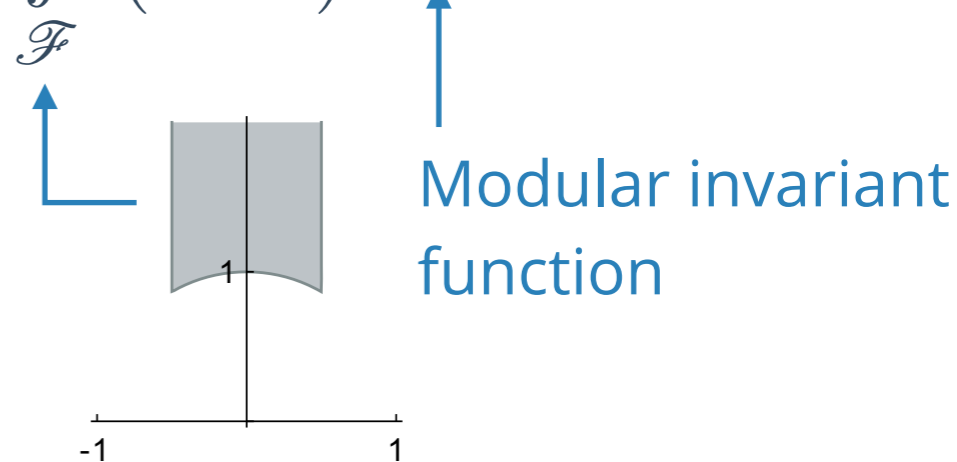
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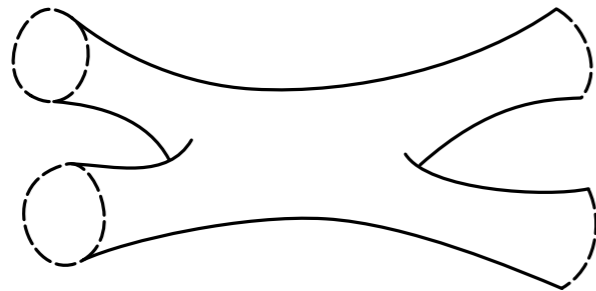


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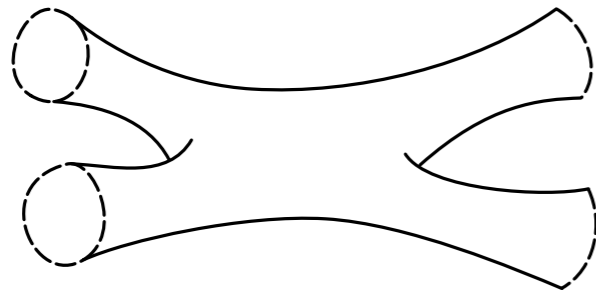
# Interactions

String theory



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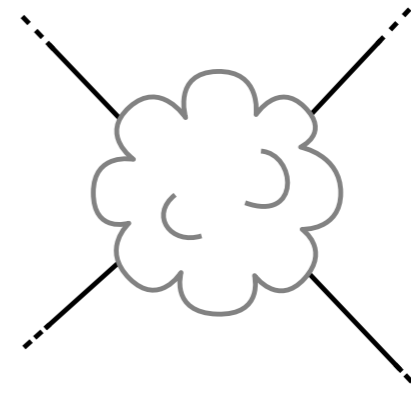
String theory



Effective field theory

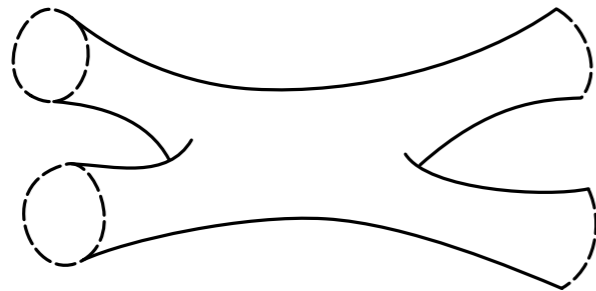


Supergravity +  $\mathcal{O}(\alpha')$



# Interactions

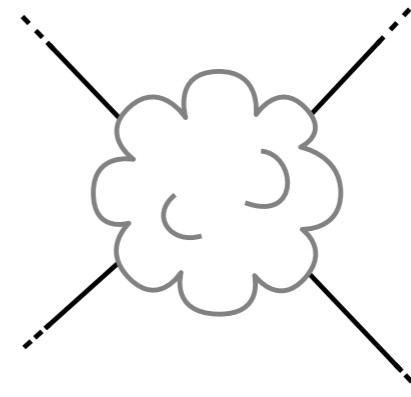
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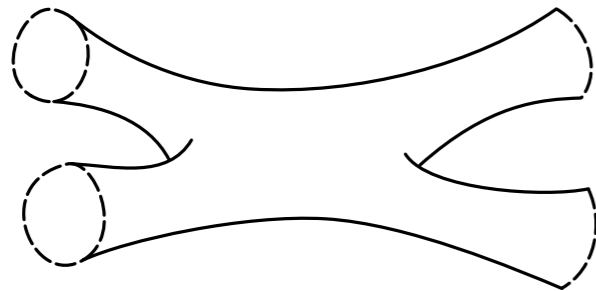
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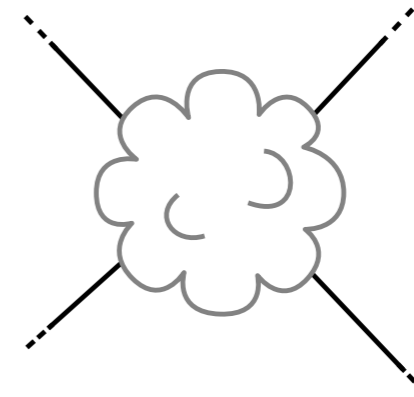
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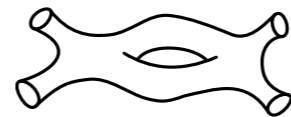
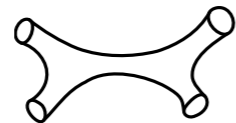
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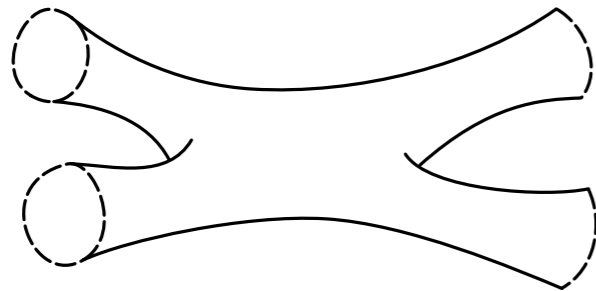


(Einstein frame)

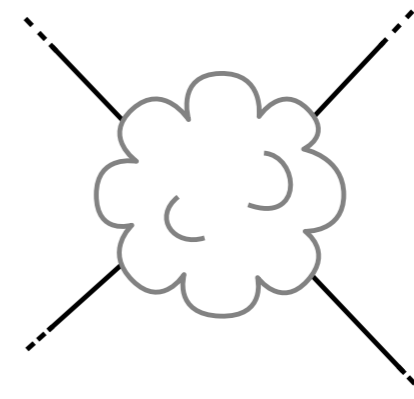
$$\mathcal{L} \propto R + (\alpha')^3 \left( 2\zeta(3)g_s^{-3/2} + 4\zeta(2)g_s^{1/2} + \dots \right) R^4 + \dots$$

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String theory



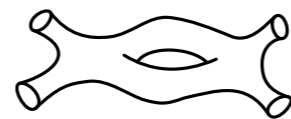
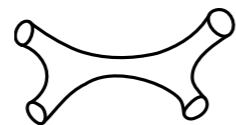
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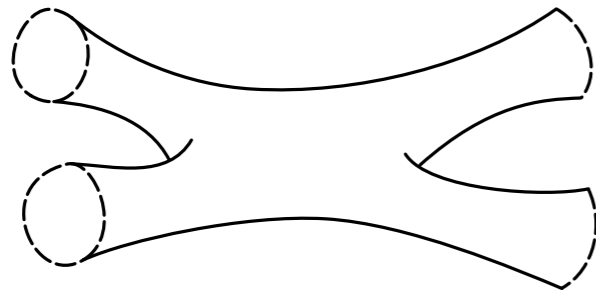
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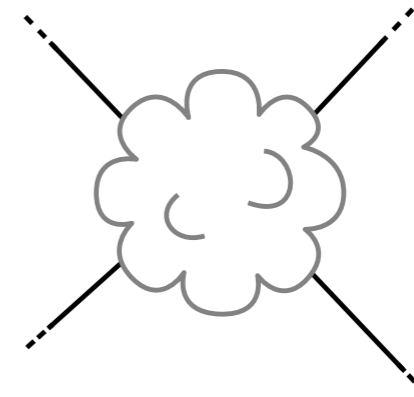


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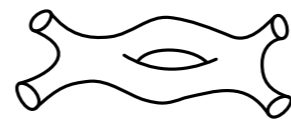
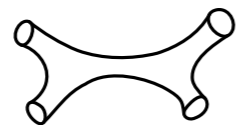
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(Einstein frame)

$$\begin{aligned} \mathcal{L} \propto R &+ (\alpha')^3 \left( 2\zeta(3)g_s^{-3/2} + 4\zeta(2)g_s^{1/2} + \dots \right) R^4 + \\ &(\alpha')^5 \left( \zeta(5)g_s^{-5/2} + \dots \right) D^4 R^4 + \\ &(\alpha')^6 \left( \frac{2}{3}\zeta(3)^2 g_s^{-3} + \frac{4}{3}\zeta(2)\zeta(3)g_s^{-1} + \dots \right) D^6 R^4 + \mathcal{O}((\alpha')^7) \end{aligned}$$

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$$\tau = \chi + ig_s^{-1}$$



# Interactions

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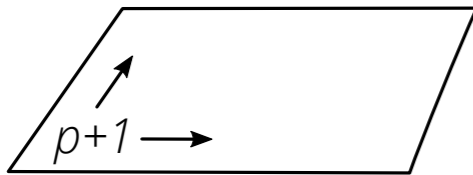
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Bao, Basu, Bossard, Cederwall, Fleig, Green, Gubay, Gutperle, HG, Kazhdan, Kiritsis, Kleinschmidt, Lambert, Miller, Nilsson, Obers, Persson, Pioline, Russo, Sethi, Vanhove, Verschinin, Waldron, West, ...

# Non-perturbative effects

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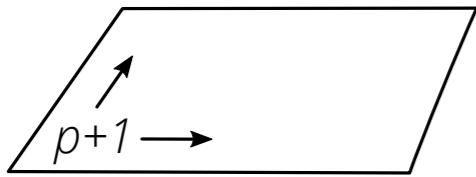


$Dp$ -brane

$p$  space directions

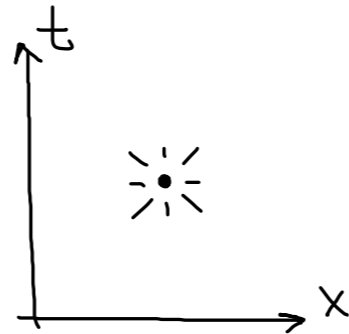
1 time direction

# Non-perturbative effects



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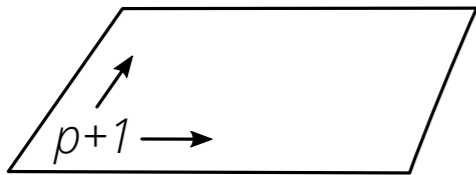
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D-instanton

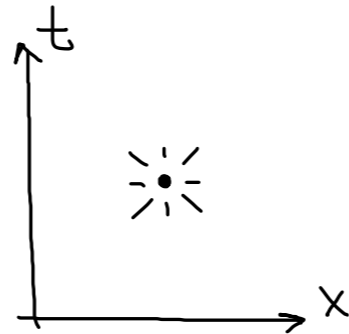
$p = -1$

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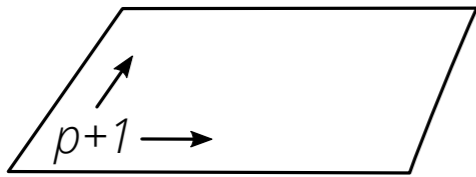
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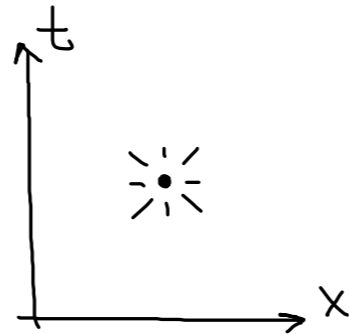


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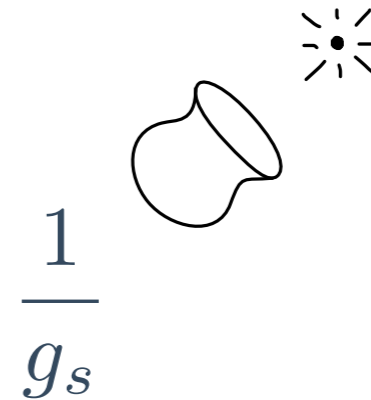
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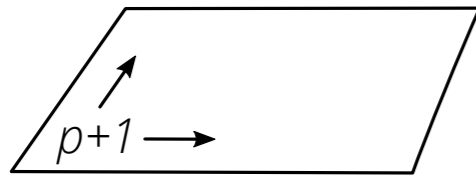


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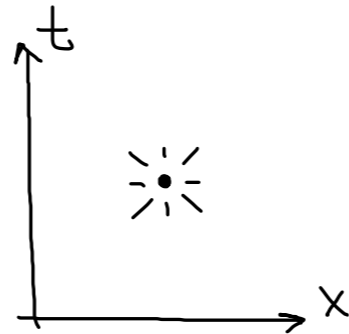


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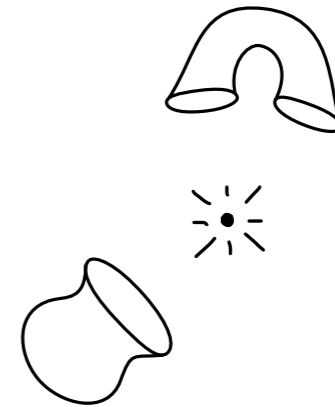
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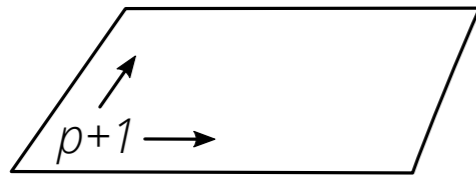
$p = -1$

$\frac{1}{g_s}$



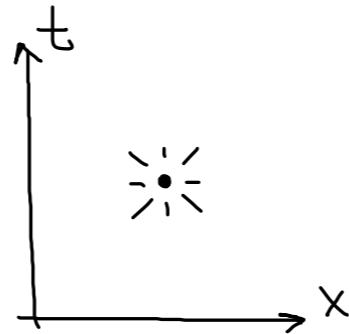
1

# Non-perturbative effects



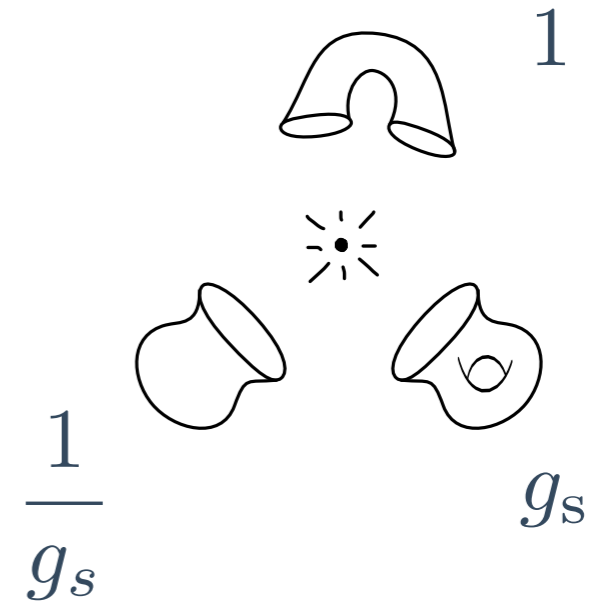
Dp-brane

$p$  space directions  
1 time direction

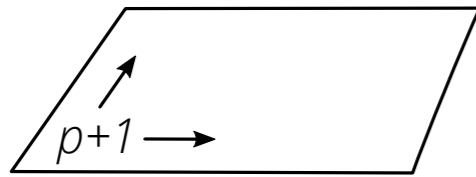


D-instanton

$p = -1$

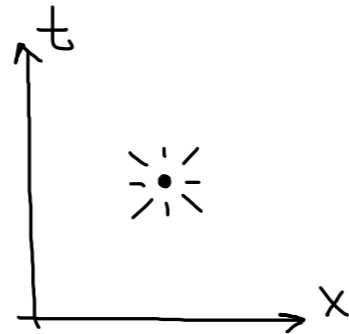


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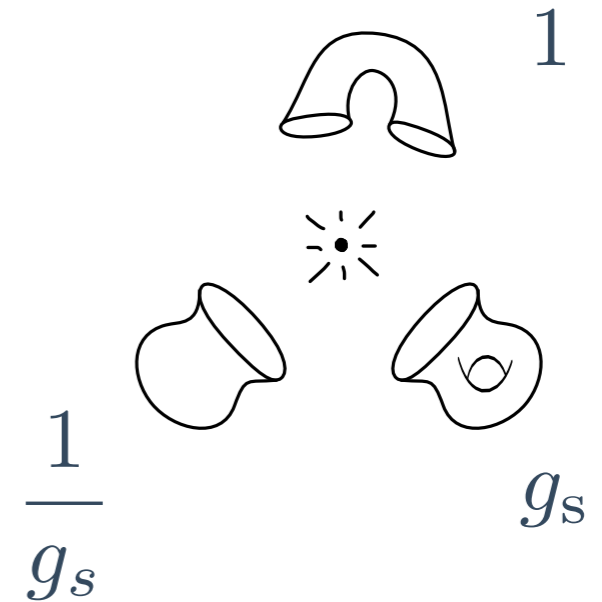


D*p*-brane

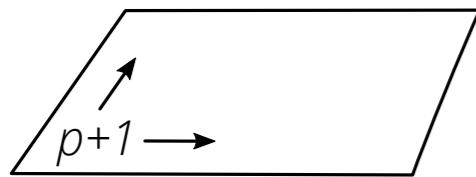
*p* space directions  
1 time direction



D-instanton  
*p* = -1

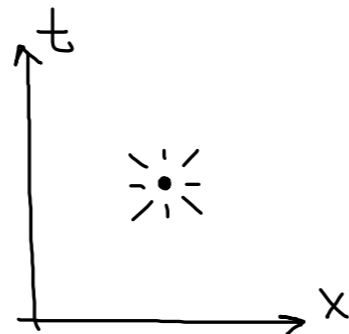


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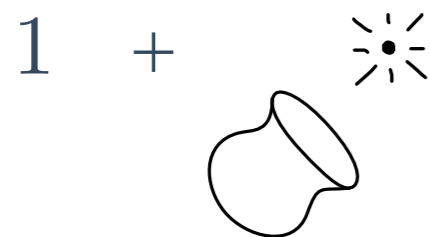
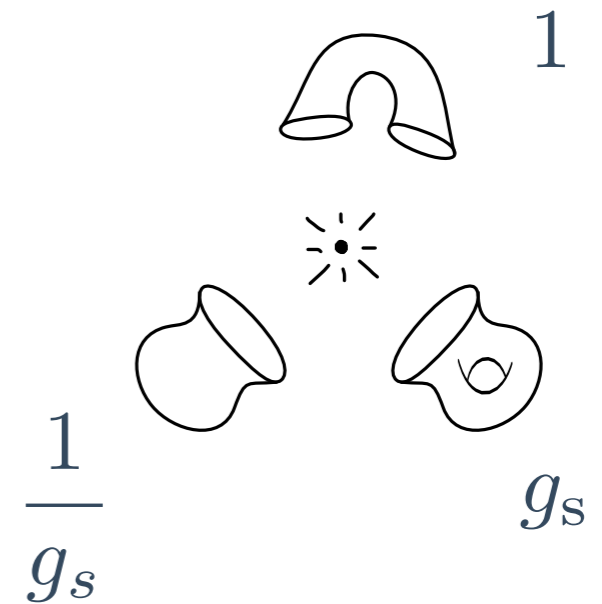


Dp-brane

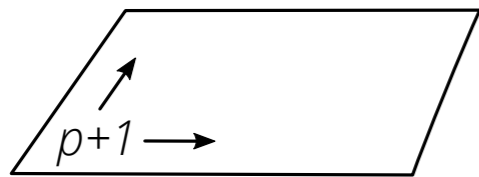
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D-instanton  
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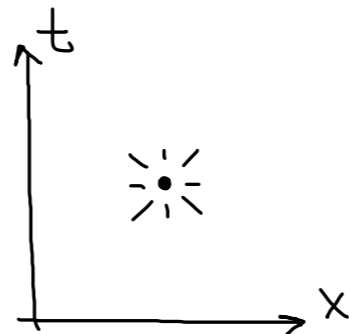


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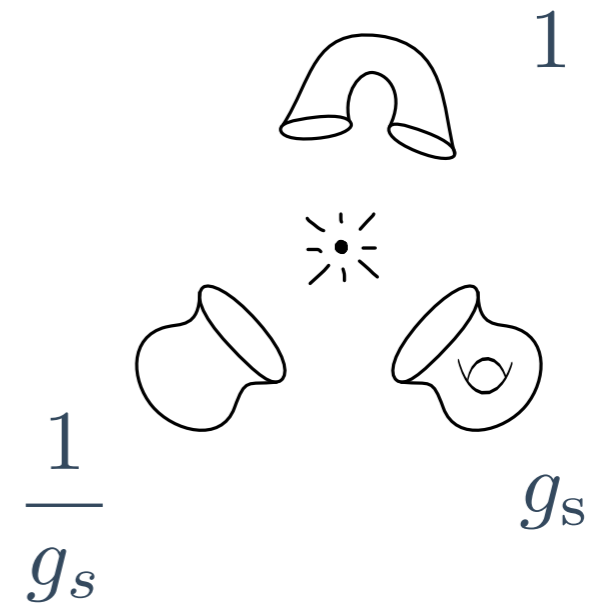


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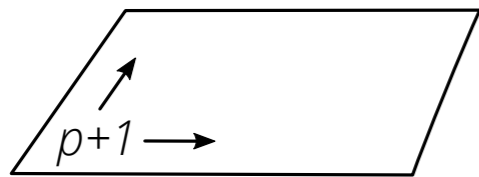


D-instanton  
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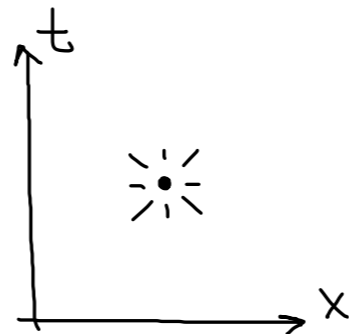
$$1 + \text{[torus with singularity]} + \frac{1}{2!} \text{[torus with singularity and another torus]}$$

# Non-perturbative effects

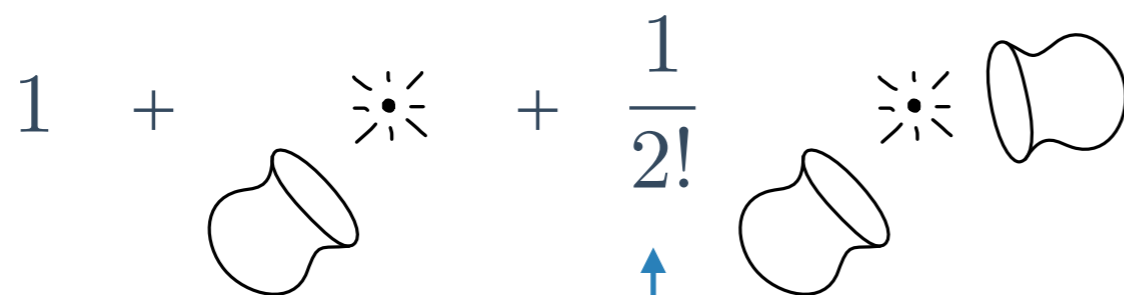
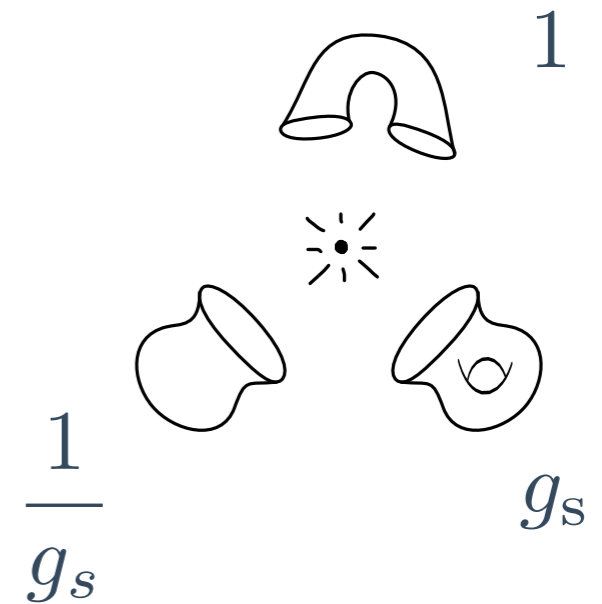


$Dp$ -brane

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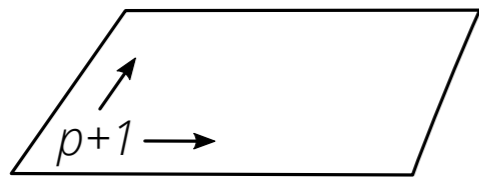


D-instanton  
 $p = -1$



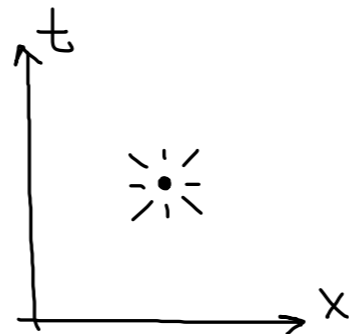
Symmetry factor for identical disks

# Non-perturbative effects

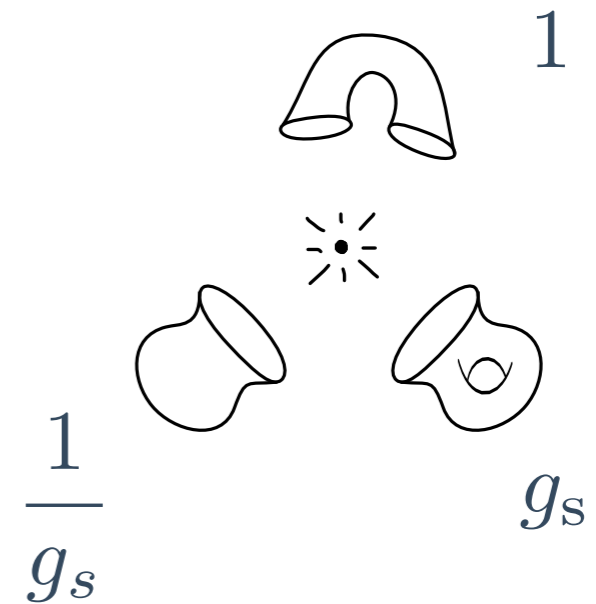


D*p*-brane

*p* space directions  
1 time direction



D-instanton  
*p* = -1



$$1 + \text{disk} + \frac{1}{2!} \text{disk} \text{ instanton} \text{ disk} + \frac{1}{3!} \text{disk} \text{ instanton} \text{ disk} \text{ disk} + \dots$$

↑ Symmetry factor for identical disks



# Non-perturbative effects

$$1 + \text{[cup]} \text{[starburst]} + \frac{1}{2!} \text{[cup]} \text{[starburst]} \text{[cup]} + \frac{1}{3!} \text{[cup]} \text{[starburst]} \text{[cup]} \text{[cup]} + \dots$$

[Green-Gutperle]

# Non-perturbative effects

$$1 + \begin{array}{c} \text{---} \\ \diagup \\ \text{---} \\ \diagdown \\ \text{---} \end{array} + \frac{1}{2!} \begin{array}{c} \text{---} \\ \diagup \\ \text{---} \\ \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \diagup \\ \text{---} \\ \diagdown \\ \text{---} \end{array} + \frac{1}{3!} \begin{array}{c} \text{---} \\ \diagup \\ \text{---} \\ \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \diagup \\ \text{---} \\ \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \diagup \\ \text{---} \\ \diagdown \\ \text{---} \end{array} + \dots$$
  
$$\exp \left( \begin{array}{c} \text{---} \\ \diagup \\ \text{---} \\ \diagdown \\ \text{---} \end{array} \right)$$

[Green-Gutperle]

# Non-perturbative effects

$$1 + \text{[one-holed torus]} \cdot \text{[sunburst]} + \frac{1}{2!} \text{[one-holed torus]} \cdot \text{[sunburst]} \cdot \text{[two-holed torus]} + \frac{1}{3!} \text{[one-holed torus]} \cdot \text{[sunburst]} \cdot \text{[two-holed torus]} \cdot \text{[one-holed torus]} + \dots$$

$$\exp\left(\text{[one-holed torus]}\right) \sim \exp\left(-\frac{\text{const}}{g_s}\right)$$

# Non-perturbative effects

$$1 + \begin{array}{c} \text{sunburst} \\ \uparrow \\ \text{torus} \end{array} + \frac{1}{2!} \begin{array}{c} \text{sunburst} \\ \uparrow \\ \text{torus} \end{array} \begin{array}{c} \text{torus} \end{array} + \frac{1}{3!} \begin{array}{c} \text{torus} \\ \uparrow \\ \text{sunburst} \end{array} \begin{array}{c} \text{torus} \end{array} + \dots$$

$$\exp\left(\begin{array}{c} \text{torus} \end{array}\right) \sim \exp\left(-\frac{\text{const}}{g_s}\right) \quad \text{Non-perturbative in } g_s$$

# Non-perturbative effects

$$1 + \begin{array}{c} \text{sunburst} \\ \uparrow \\ \text{torus} \end{array} + \frac{1}{2!} \begin{array}{c} \text{sunburst} \\ \uparrow \\ \text{torus} \end{array} \begin{array}{c} \text{sunburst} \\ \uparrow \\ \text{torus} \end{array} + \frac{1}{3!} \begin{array}{c} \text{sunburst} \\ \uparrow \\ \text{torus} \end{array} \begin{array}{c} \text{sunburst} \\ \uparrow \\ \text{torus} \end{array} + \dots$$

$$\exp\left(\begin{array}{c} \text{sunburst} \\ \uparrow \\ \text{torus} \end{array}\right) \sim \exp\left(-\frac{\text{const}}{g_s}\right) \quad \text{Non-perturbative in } g_s$$

$$\mathcal{E}_0(\tau) = 2\zeta(3)\tau_2^{3/2} + 4\zeta(2)\tau_2^{-1/2} + \dots + \underbrace{C e^{2\pi i\tau}}_{\text{.....}} + \dots$$

$$\tau = \tau_1 + i\tau_2 = \chi + ig_s^{-1}$$

[Green-Gutperle]

# Moduli space

$$R + (\alpha')^3 \mathcal{E}_0^{(D)}(g) R^4 + (\alpha')^5 \mathcal{E}_4^{(D)}(g) D^4 R^4 + (\alpha')^6 \mathcal{E}_6^{(D)}(g) D^6 R^4 + \dots$$

$$\mathcal{M}_{\text{classical}} = G(\mathbb{R})/K$$

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$$\mathcal{M}_{\text{classical}} = G(\mathbb{R})/K$$

$D$	$G(\mathbb{R})$	$K$
10	$SL(2, \mathbb{R})$	$SO(2)$
9	$SL(2, \mathbb{R}) \times \mathbb{R}^+$	$SO(2)$
8	$SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$	$SO(3) \times SO(2)$
7	$SL(5, \mathbb{R})$	$SO(5)$
6	$Spin(5, 5; \mathbb{R})$	$(Spin(5) \times Spin(5))/\mathbb{Z}_2$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$
4	$E_7(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$
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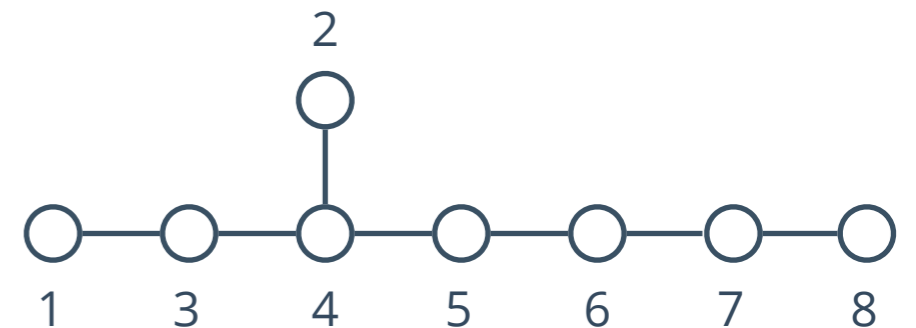
[Cremmer-Julia]

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[Cremmer-Julia]



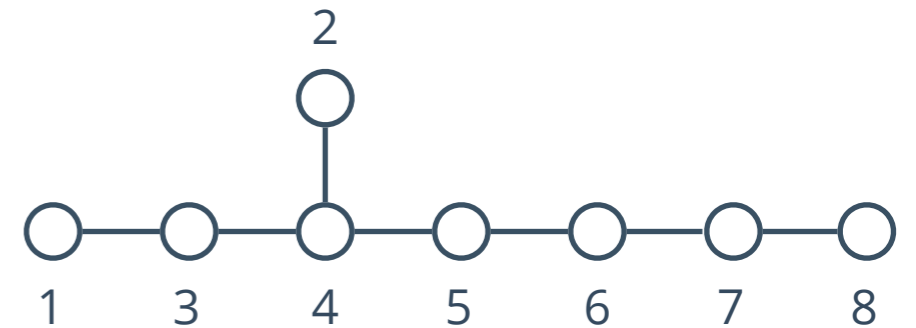
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$$\mathcal{M}_{\text{classical}} = G(\mathbb{R})/K$$

$$\mathcal{E}_n(\tau) = \mathcal{E}_n^{(10)}(g)$$

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9	$SL(2, \mathbb{R}) \times \mathbb{R}^+$	$SO(2)$
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# Moduli space

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$$\tau = \chi + ig_s^{-1} \in \mathbb{H} = \{z \in \mathbb{C} \mid \text{Im } z > 0\} \cong SL(2, \mathbb{R})/SO(2, \mathbb{R})$$

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No similar structure for lower dimensions

# U-duality

$G(\mathbb{R}) \curvearrowright \mathcal{M}_{\text{classical}}$  classical symmetry

[Hull-Townsend]

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Quantization of charges

[Hull-Townsend]

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Quantization of charges  $\implies$  classical symmetry  $\longrightarrow$  discrete symmetry

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All observables are invariant under  $G(\mathbb{Z})$

[Hull-Townsend]

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$$\mathcal{E}_0^{(D)}(g), \mathcal{E}_4^{(D)}(g), \mathcal{E}_6^{(D)}(g) : G(\mathbb{Z}) \backslash G(\mathbb{R}) / K \rightarrow \mathbb{C}$$

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- (C)  $\varphi$  is an eigenfunction to all  $G$ -invariant differential operators



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- (C) Z-finiteness:  $\dim(\text{span}\{X\varphi(g) \mid X \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})\}) < \infty$
- (D) Growth: for any norm  $\|\cdot\|$  on  $G(\mathbb{R})$  there exists a positive integer  $n$  and constant  $C$  such that  $|\varphi(g)| \leq C\|g\|^n$

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An *automorphic form* is a smooth function  $\varphi : G(\mathbb{R}) \rightarrow \mathbb{C}$  satisfying the following conditions

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- (B) K-finiteness:
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- (D) Growth:

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- (A) Automorphic invariance: ✓ U-duality
- (B) K-finiteness: ✓ spherical
- (C) Z-finiteness:
- (D) Growth: ✓ weak coupling limit from string perturbation theory

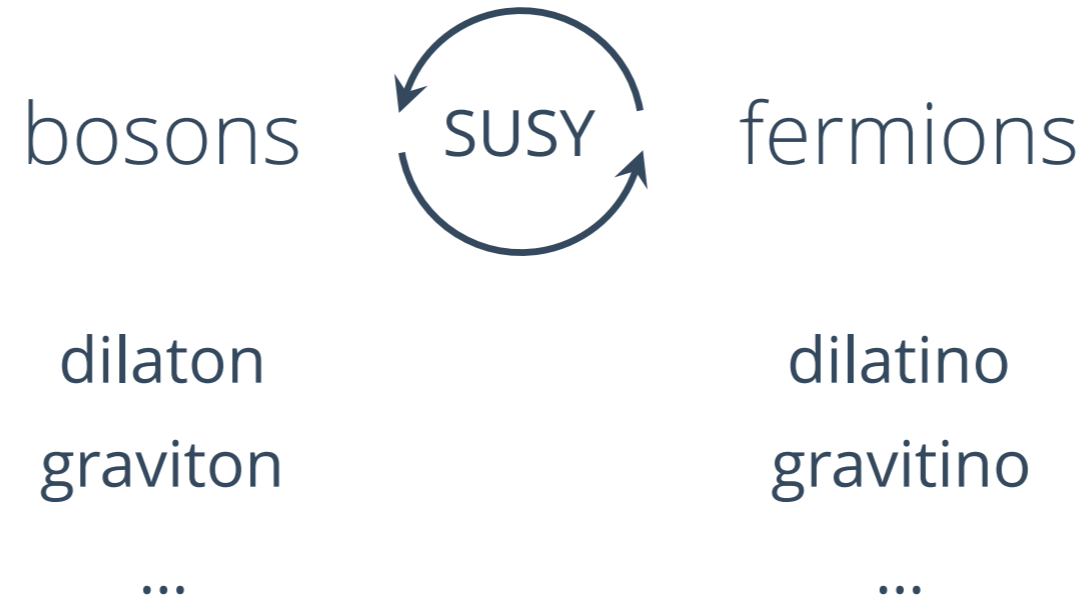


# Automorphic forms

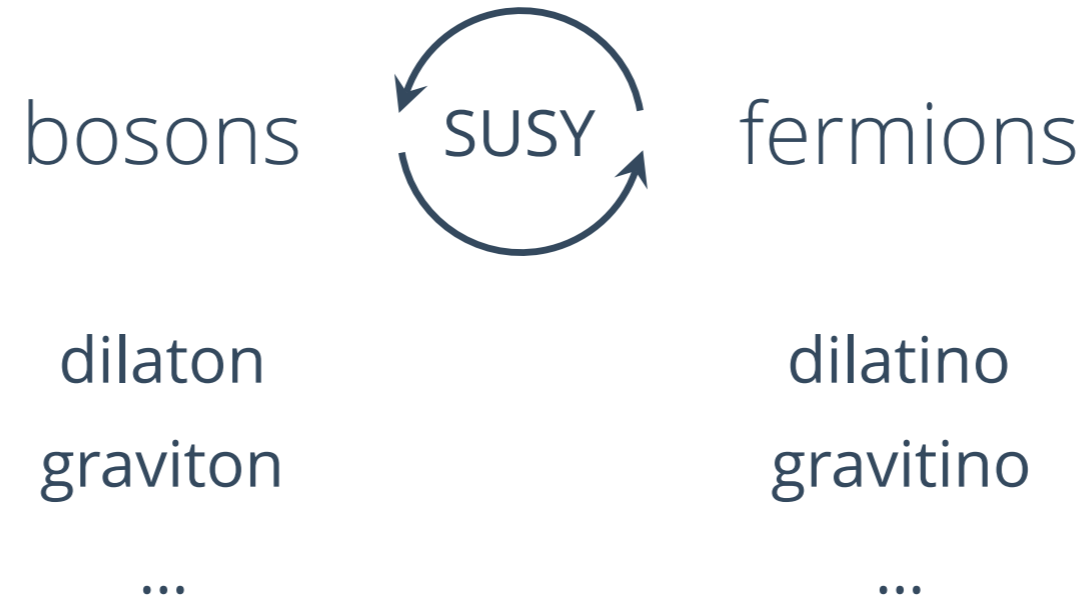
An *automorphic form* is a smooth function  $\varphi : G(\mathbb{R}) \rightarrow \mathbb{C}$  satisfying the following conditions

- (A) Automorphic invariance: ✓ U-duality
- (B) K-finiteness: ✓ spherical
- (C) Z-finiteness: ?
- (D) Growth: ✓ weak coupling limit from string perturbation theory

# Supersymmetry constraints



# Supersymmetry constraints



10 dimensions:

# Supersymmetry constraints



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$$\mathcal{L}^{(3)} =$$

$$\mathcal{E}_0(\tau) R^4$$

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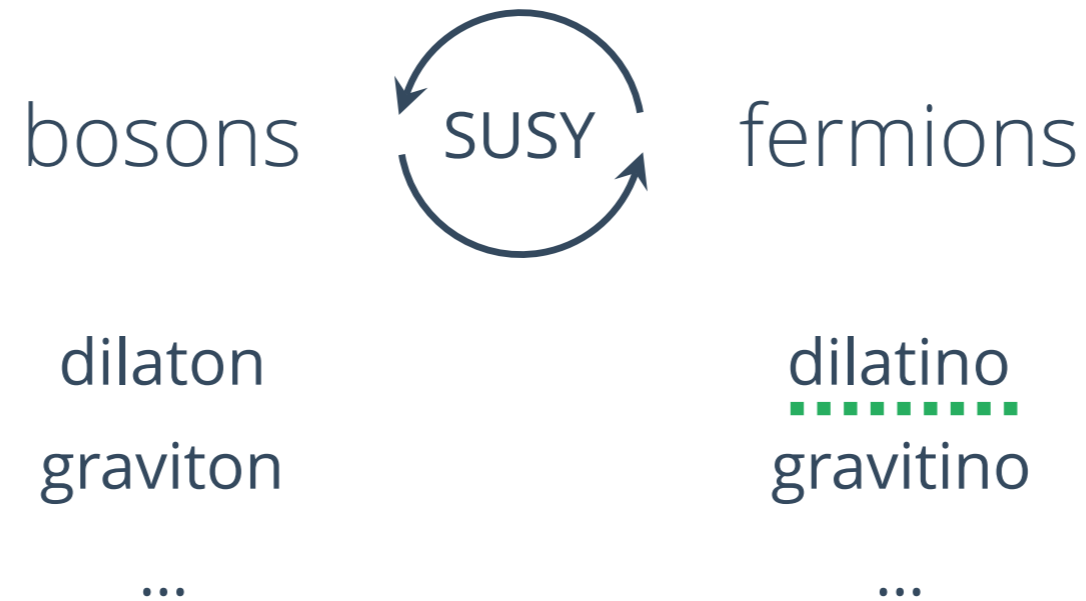


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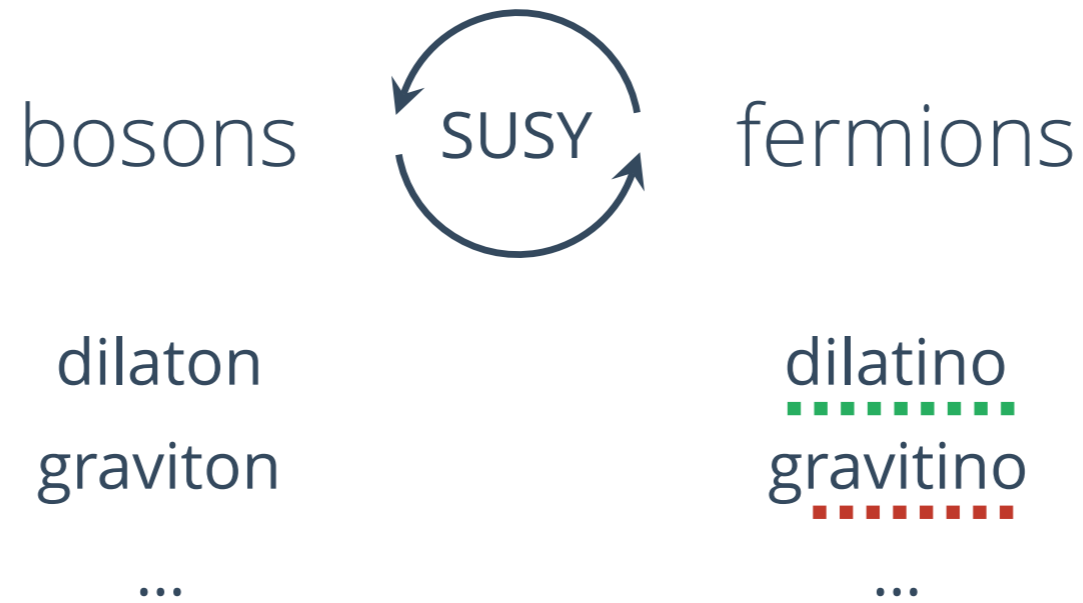
# Supersymmetry constraints



10 dimensions:

$$\mathcal{L}^{(3)} = f_{12}(\tau)\lambda^{\dots 16} + f_{11}(\tau)\hat{G}\lambda^{14} + \dots + f_0(\tau)R^4 + \dots + f_{-12}(\tau)\lambda^{\dots *16}$$

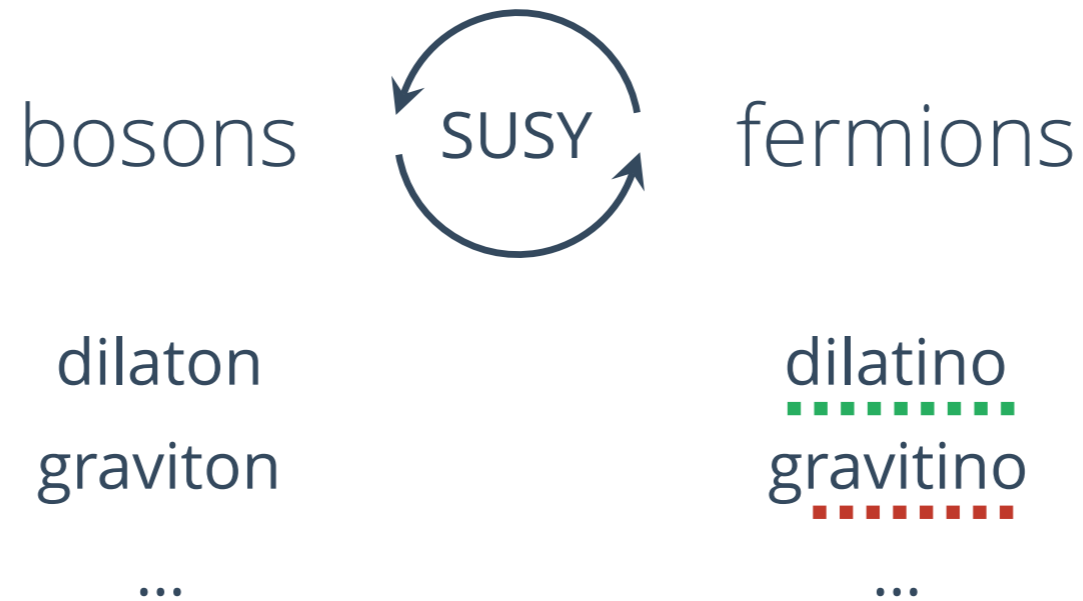
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Linearized SUSY:  $f_{w+1}(\tau) = i\left(\tau_2 \frac{\partial}{\partial \tau} - i\frac{w}{2}\right) f_w(\tau)$

[Green-Sethi]



# Supersymmetry constraints

$$\int d^D x \sqrt{-g} \mathcal{L} = S = S^{(0)}$$
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$$\int d^D x \sqrt{-g} \mathcal{L} = S = S^{(0)} + (\alpha')^3 S^{(3)} + (\alpha')^5 S^{(5)} + \dots$$
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Similarly for lower dimensions

# Eisenstein series

$$E(s; \tau) =$$

$$s \in \mathbb{C}$$



# Eisenstein series

$$E(s; \tau) = \sum_{\substack{c, d \in \mathbb{Z} \\ (c, d) \neq (0, 0)}} \frac{\text{Im}(\tau)^s}{|c\tau + d|^{2s}}$$

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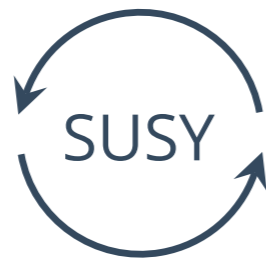
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[Green-Gutperle, Pioline, Green-Russo-Vanhove]

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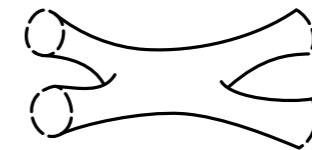
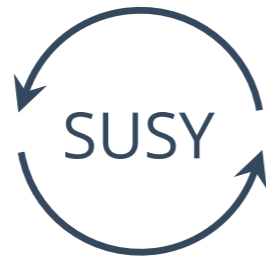
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(Einstein frame)

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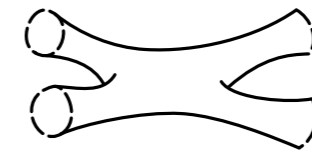
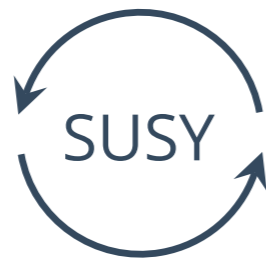
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[Green-Gutperle, Pioline, Green-Russo-Vanhove]

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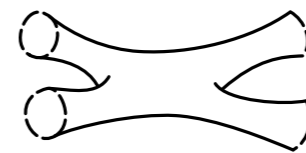
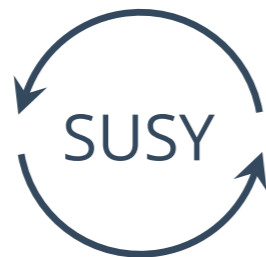
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[Green-Gutperle, Pioline, Green-Russo-Vanhove]

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$\mathcal{E}_6(\tau)$  as a sum over images  $\sum_{B(\mathbb{Q}) \backslash G(\mathbb{Z})}$  but not of a character  $\chi$

[Green-Miller-Vanhove]

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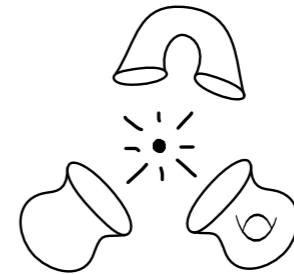
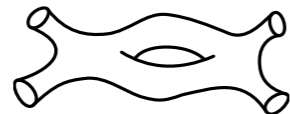
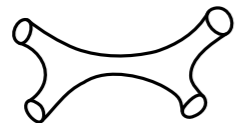
Perturbative  
(zero-mode)

Non-perturbative  
(remaining modes)

# Extracting physical information

Expand Bessel function in  $g_s$

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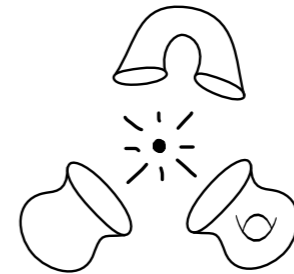
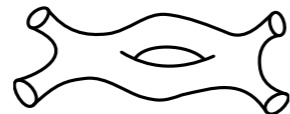
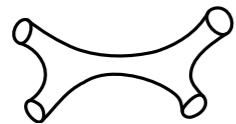
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wrapping number and charge  
of a T-dual D-particle

[Green-Gutperle]

# Lower dimensions

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$D$	$G(\mathbb{R})$	$K$	$G(\mathbb{Z})$
10	$SL(2, \mathbb{R})$	$SO(2)$	$SL(2, \mathbb{Z})$
9	$SL(2, \mathbb{R}) \times \mathbb{R}^+$	$SO(2)$	$SL(2, \mathbb{Z}) \times \mathbb{Z}_2$
8	$SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$	$SO(3) \times SO(2)$	$SL(3, \mathbb{Z}) \times SL(2, \mathbb{Z})$
7	$SL(5, \mathbb{R})$	$SO(5)$	$SL(5, \mathbb{Z})$
6	$Spin(5, 5; \mathbb{R})$	$(Spin(5) \times Spin(5)) / \mathbb{Z}_2$	$Spin(5, 5; \mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp(8) / \mathbb{Z}_2$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU(8) / \mathbb{Z}_2$	$E_7(\mathbb{Z})$
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$$E(\chi; g) = \sum_{\gamma \in B(\mathbb{Z}) \setminus G(\mathbb{Z})} \chi(\gamma g)$$

# Parabolic subgroups

Fourier expand  
in different directions



Unipotent subgroup  $U$

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Positive roots

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$$P = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & * & * & * \\ & & * & * \end{pmatrix} \right\} = LU$$

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Maximal parabolic

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Minimal parabolic  
Borel



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Minimal parabolic  
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$$B = NA$$

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Maximal parabolic

$$P = LU$$

$$U = \left\{ \begin{pmatrix} \boxed{1} & & & * \\ & \boxed{1} & & * \\ & & \boxed{1} & * \\ & & & \boxed{1} \end{pmatrix} \right\}$$



# Fourier expansion

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Let  $\psi : U(\mathbb{Z}) \backslash U(\mathbb{R}) \rightarrow U(1)$  be a multiplicative character

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$$E(\chi; g) = \sum_{\psi} F_U(\chi, \psi; g)$$



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$$U^{(1)} = U \quad U^{(n+1)} = [U^{(n)}, U^{(n)}]$$

# Terminology

$P = B \longrightarrow U = N$  Fourier coefficient is a Whittaker coefficient

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Characters and coefficients with all  $m_\alpha \neq 0$  are called **generic**  
otherwise they are called **degenerate**

# Fourier expansion

Choice of unipotent subgroup  $U$   $\longleftrightarrow$  Study different perturbative and non-perturbative effects

[Green-Miller-Vanhove]

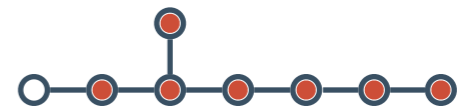


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Choice of unipotent subgroup  $U$   $\longleftrightarrow$  Study different perturbative and non-perturbative effects

- String perturbation limit  
D-instantons | NS5-instantons

$$g_s \rightarrow 0$$



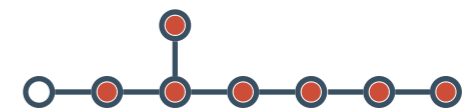
[Green-Miller-Vanhove]

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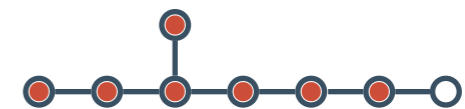
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- Decompactification limit  
Higher dimensional black holes | BPS states

Large radius for  
compactified circle



[Green-Miller-Vanhove]

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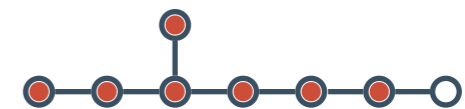
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- M-theory limit  
M2, M5-instantons

Large M-theory torus



[Green-Miller-Vanhove]

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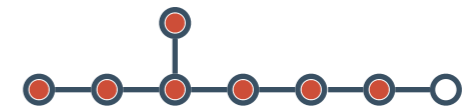
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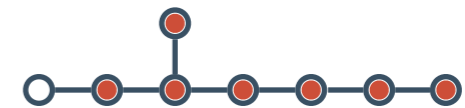
Maximal parabolic  
subgroups

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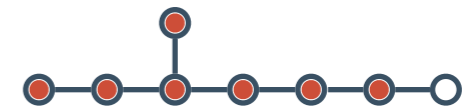
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[Green-Miller-Vanhove]

Maximal parabolic subgroups

Difficult to compute!

# Fourier expansion

Goal: find expressions for Fourier coefficients  
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# Adelic framework

*An **efficient**, but abstract, way to approach the subject of automorphic forms is by the introduction of **adeles**, rather **ungainly objects** that nevertheless, once familiar, **spare** much unnecessary thought and **many useless calculations**.*

— Robert P. Langlands\*

\*Representation theory - its rise and its role in number theory, Proceedings of the Gibbs symposium (1989)



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*An **efficient**, but abstract, way to approach the subject of automorphic forms is by the introduction of **adeles**, rather **ungainly objects** that nevertheless, once familiar, **spare** much unnecessary thought and **many useless calculations**.*

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Adelic Eisenstein series



Eisenstein series

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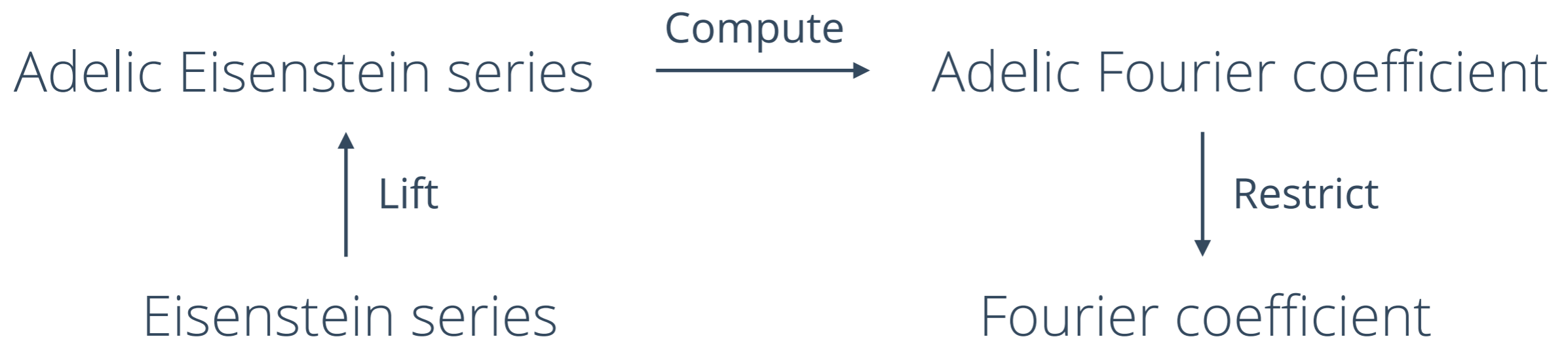


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# The adèles

For a prime  $p$

$\mathbb{Q}$

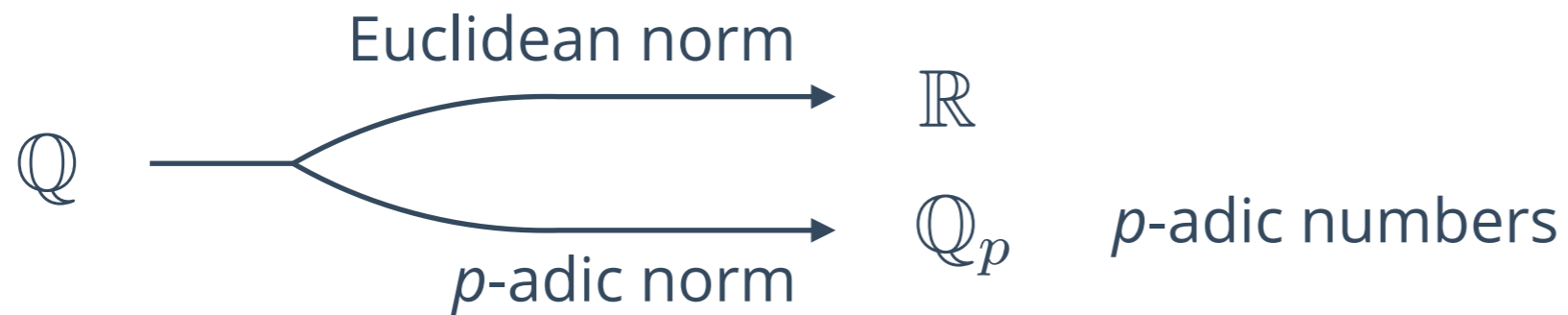
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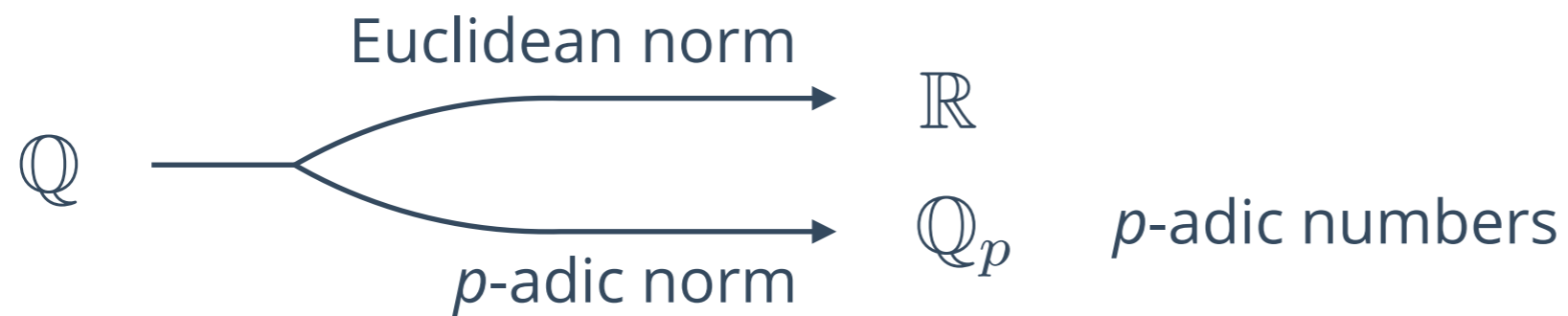
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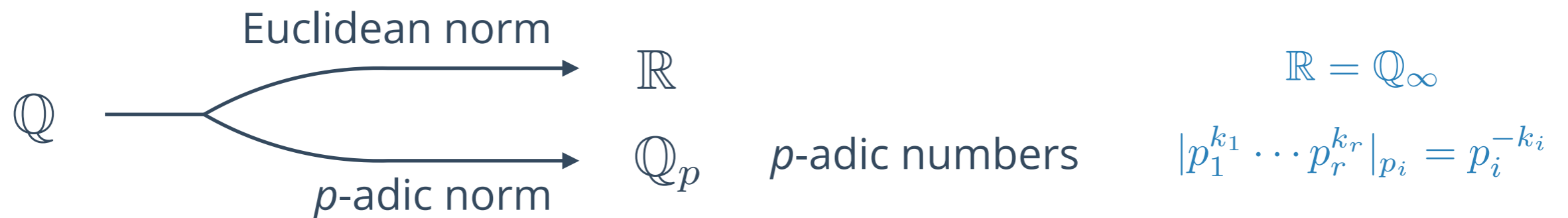


$$\mathbb{R} = \mathbb{Q}_\infty$$



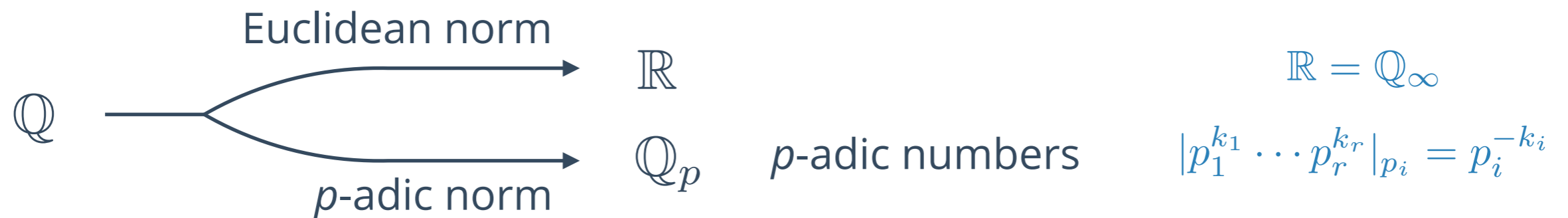
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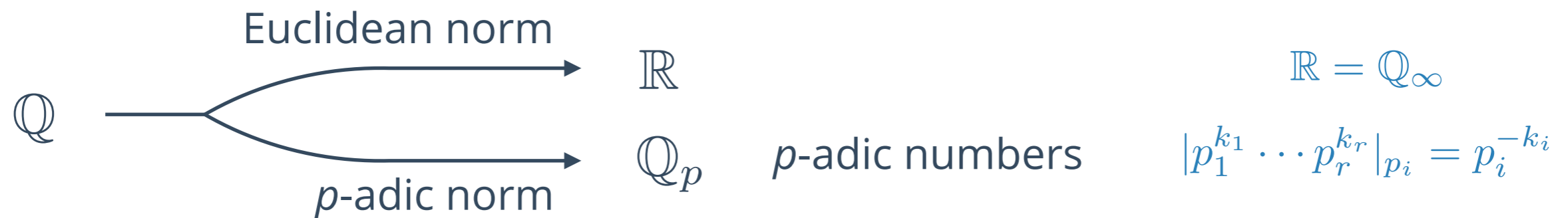


The adeles are then defined as

$$\mathbb{A} = \mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \prod'_{p \text{ prime}} \mathbb{Q}_p \quad x = (x_\infty; x_2, x_3, x_5, \dots) \in \mathbb{A}$$

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$$\mathbb{Q} \hookrightarrow \mathbb{A}$$

$$q \mapsto (q; q, q, \dots)$$

$\mathbb{Q}$  is discrete in  $\mathbb{A}$  taking the role of  $\mathbb{Z}$  in  $\mathbb{R}$

Much easier to work with since it is a field!

# Adelic framework

$$\mathcal{E}_0^{(D)}(g), \mathcal{E}_4^{(D)}(g), \mathcal{E}_6^{(D)}(g) : G(\mathbb{Z}) \backslash G(\mathbb{R}) / K \rightarrow \mathbb{C}$$

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[FGKP15 §4.2.2]

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Fourier coefficients  $\longrightarrow$  Adelic Fourier coefficients

$$\int_{U(\mathbb{Z}) \backslash U(\mathbb{R})} E(\chi; ug) \overline{\psi_{\mathbb{R}}(u)} du$$

$$\int_{U(\mathbb{Q}) \backslash U(\mathbb{A})} E(\chi; ug) \overline{\psi_{\mathbb{A}}(u)} du$$

$$m_{\alpha} \in \mathbb{Z}$$

$$m_{\alpha} \in \mathbb{Q}$$

# Computing adelic Fourier coefficients

[FGKP15 §9-10]

Whittaker coefficients

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Factorisation

[GKP14]

Fourier coefficients

In terms of Whittaker coefficients

Simplify drastically for certain  $\chi$

# Example of simplifications

$$G = SL(3)$$

$$E(\chi; g)$$

$$\chi \longleftrightarrow (s_1, s_2) \in \mathbb{C}^2$$

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$$N = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}$$

$$\psi_N \left( \begin{pmatrix} 1 & x_1 & * \\ & 1 & x_2 \\ & & 1 \end{pmatrix} \right) = e^{2\pi i(m_1 x_1 + m_2 x_2)}$$

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$$W_N(\chi, \psi_{m_1, m_2}; g) \propto \left( \begin{matrix} \text{arithmetic} \\ \text{factor} \end{matrix} \right) \int K_{\#}(\dots) K_{\#}(\dots)$$

[FGKP15 §10.6]

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Vanishes for certain  $(s_1, s_2)$

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# Example of simplifications

Certain  $(s_1, s_2)$

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# Automorphic representations

$G(\mathbb{A}) \curvearrowright$  Space of automorphic forms\*

\* With some subtleties described in [FGKP15 §6]

[Bump, Goldfeld-Hundley]

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What is a small automorphic representation?

\* With some subtleties described in [FGKP15 §6]

[Bump, Goldfeld-Hundley]

# Wavefront set

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg,  
Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]



# Wavefront set

The (global) wavefront set contains all the characters  $\psi$  which can give rise to non-vanishing Fourier coefficients in that representation

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Small automorphic representations have few non-vanishing Fourier coefficients

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

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Characters  $\psi$   $\longleftrightarrow$  Nilpotent elements in  $\mathfrak{g}(\mathbb{Q})$

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Closure with respect to partial ordering

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 decreasing order  
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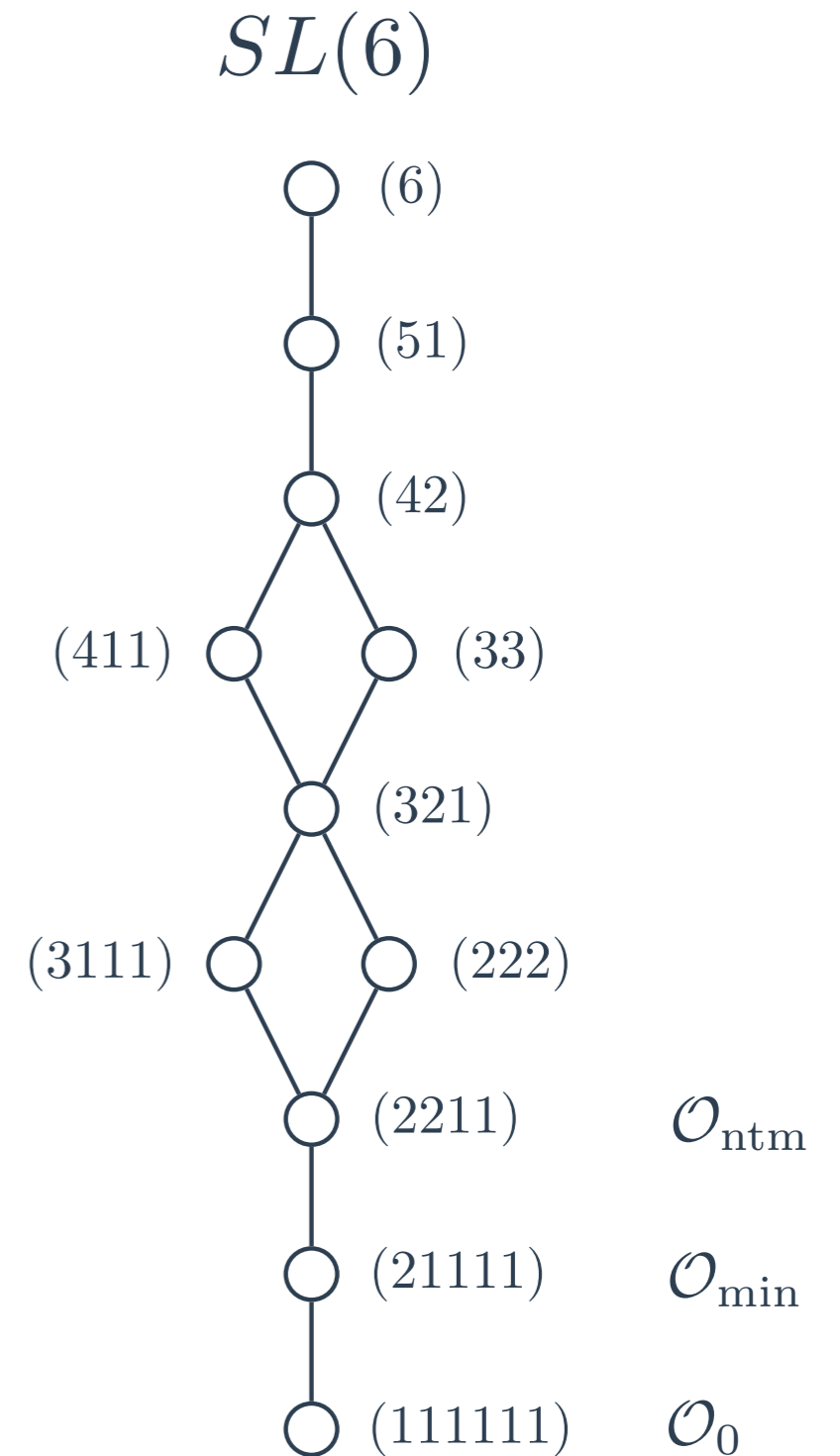
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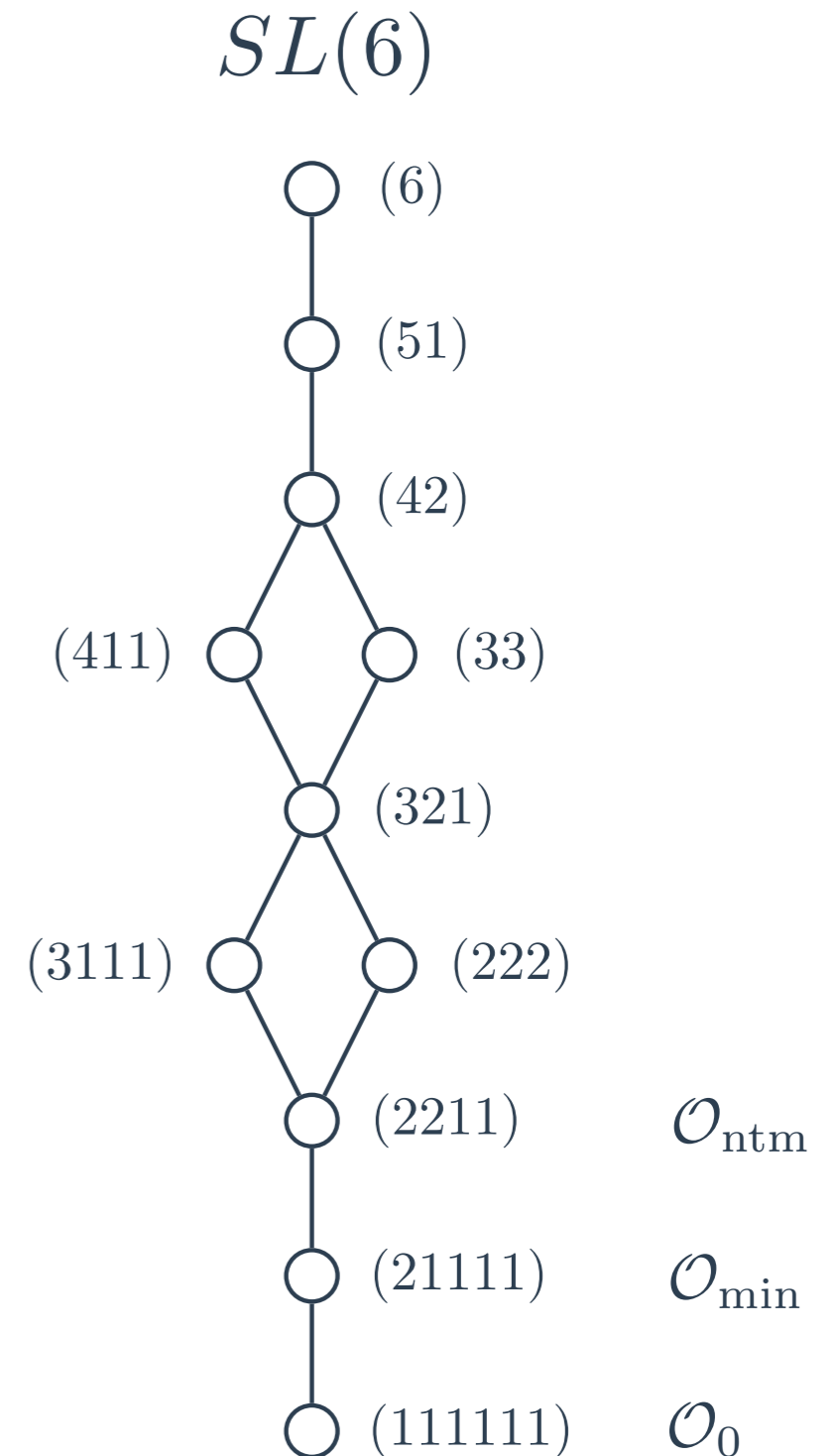
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Illustrated by a Hasse diagram

Closure:  $\overline{\mathcal{O}} = \bigcup_{\mathcal{O}' \leq \mathcal{O}} \mathcal{O}'$





# Automorphic representations

Small representations

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$$\mathrm{WF}(\pi_{\min}) = \overline{\mathcal{O}_{\min}} = \mathcal{O}_{\min} \cup \mathcal{O}_0$$

$$\mathrm{WF}(\pi_{\mathrm{ntm}}) = \overline{\mathcal{O}_{\mathrm{ntm}}} = \mathcal{O}_{\mathrm{ntm}} \cup \mathcal{O}_{\min} \cup \mathcal{O}_0$$

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$$\mathcal{E}_0^{(D)} \in \pi_{\min} \quad \mathcal{E}_4^{(D)} \in \pi_{\mathrm{ntm}}$$

[Green-Miller-Vanhove,  
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$$\int_K K \longrightarrow 0$$

$$\sum_K K \longrightarrow K$$

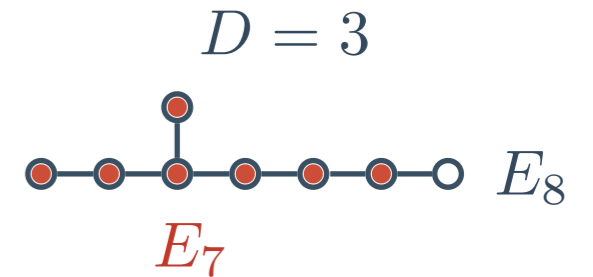
# Automorphic representations

[Ferrara-Günaydin, Ferrara-Maldacena, Green-Miller-Vanhove]

# Automorphic representations

- Decompactification limit  
Higher dimensional black holes | BPS states

Large radius for  
compactified circle



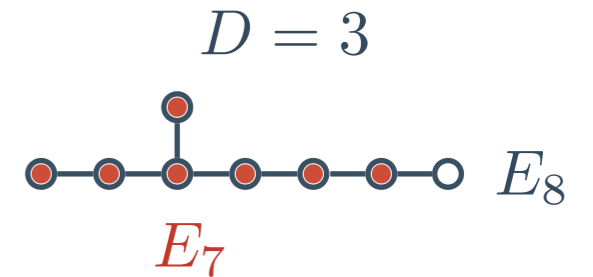
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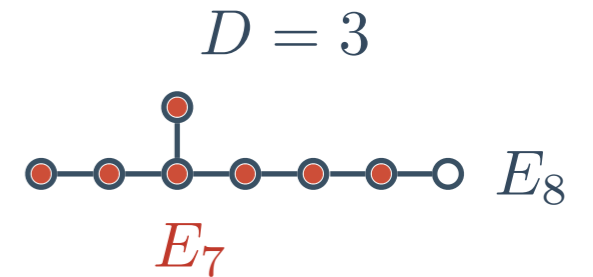
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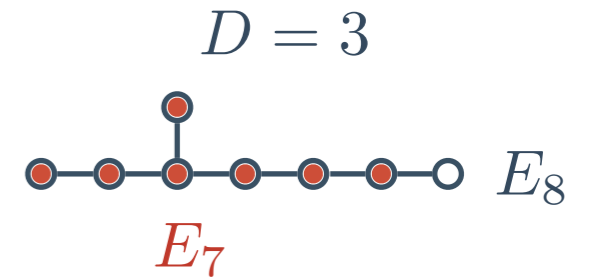
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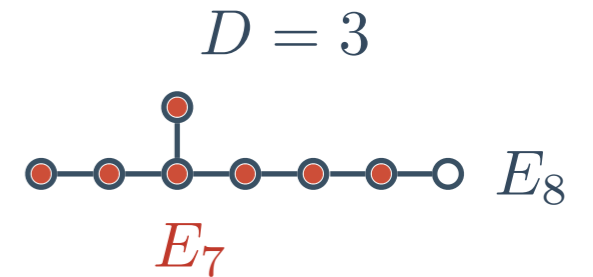
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	$\pi_{\min}$	$\pi_{\text{ntm}}$	$\pi_{3A_1}$	$\pi_{A_2}$	
$\dim\{\psi_U \in \text{WF}\}$	28	45	55	56	[Miller-Sahi]

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Higher dimensional black holes | BPS states

Large radius for compactified circle



	$\mathcal{E}_0^{(D)}$	$\mathcal{E}_4^{(D)}$			
	$\pi_{\min}$	$\pi_{\text{ntm}}$	$\pi_{3A_1}$	$\pi_{A_2}$	
$\dim\{\psi_U \in \text{WF}\}$	28	45	55	56	[Miller-Sahi]
$D = 4$ BPS-orbits $E_7 \curvearrowright \{\psi_U\}$	$\frac{1}{2}$ BPS	$\frac{1}{4}$ BPS	$\frac{1}{8}$ BPS	$\frac{1}{8}$ BPS <sup>+</sup>	

[Ferrara-Günaydin, Ferrara-Maldacena, Green-Miller-Vanhove]

Goal: find expressions for Fourier coefficients  
in terms of (known) Whittaker coefficients  
using vanishing properties of the given  $\pi$

# Previous results

[Miller-Sahi]

# Previous results

## Theorem

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For  $G = E_6, E_7$ , an automorphic form  $\varphi \in \pi_{\min}$  is completely determined by maximally degenerate Whittaker coefficients

$W_N$  with only one  $m_\alpha \neq 0$

[Miller-Sahi]

# Main results

*SL(3), SL(4)*

[GKP14]



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[GKP14]

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$$\varphi = \sum_{\mathcal{O}} \varphi_{\mathcal{O}} \quad \text{where } \varphi_{\mathcal{O}} \text{ vanishes unless } \mathcal{O} \subseteq \text{WF}(\pi)$$

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$\varphi \in \pi_{\min}$  maximally degenerate Whittaker coefficients

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$\varphi \in \pi_{\min}$     maximally degenerate Whittaker coefficients    single root

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single root

$$\varphi \in \pi_{\text{ntm}}$$

at most two commuting roots

[GKP14]

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Fourier coefficients on maximal parabolic subgroups in the minimal representation



$\pi_{\min}$

[GKP14]

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$$F_U(\chi_{\min}, \psi; g) = W_N(\chi_{\min}, \psi'; lg) \quad \text{with } l \in L(\mathbb{Q}) \text{ depending on } \psi$$

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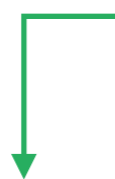
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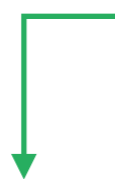
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Maximally degenerate

[GKP14]

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$$\int_{U(\mathbb{Q}) \backslash U(\mathbb{A})} E(\chi_{\min}; ug) \overline{\psi_U(u)} \, du = \int_{N(\mathbb{Q}) \backslash N(\mathbb{A})} E(\chi_{\min}; nlg) \overline{\psi'_N(n)} \, dn$$

# Other groups

[Work in progress with Ahlén, Liu, Kleinschmidt, Persson]

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$$SL(n)$$

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and similar statement for next-to-minimal representation

[Work in progress with Ahlén, Liu, Kleinschmidt, Persson]



# Other groups

## Conjecture

---

A similar relations holds for all simple Lie groups

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[GKP14]

[Proof in progress with Gourevitch, Kleinschmidt, Persson, Sahi]

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Would allow us to compute non-perturbative effects that capture information about BPS-states and black holes

[GKP14]

[Proof in progress with Gourevitch, Kleinschmidt, Persson, Sahi]

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- Simplification of Fourier coefficients with  $\chi_{\min}$  for dimensions lower than three. Kac-Moody groups  $E_9, E_{10}, E_{11}$   
[Fleig-Kleinschmidt, Fleig-Kleinschmidt-Persson]

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[Fleig-Kleinschmidt, Fleig-Kleinschmidt-Persson]

How to define “small automorphic representations” for Kac-Moody groups? What is the mechanism behind the vanishing properties?

- $\mathcal{E}_6 D^6 R^4$  requires extended notion of automorphic forms, the development of which will positively bring new exciting insights to both physics and mathematics.

# Thank you!

Henrik Gustafsson

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